October 8: Week 6 - Proof Techniques

Techniques: Invariance/Monovariance, Induction, Pigeon Hole Principle

Here are some problems from previous weeks that illustrate the above techniques well.

Strong Induction

(Week 5) Suppose that x is a real number such that $x+\frac{1}{x}$ is an integer. Prove that $x^n+\frac{1}{x^n}$ is an integer for every integer n.

Induction by cases

(Week 3) Show that every positive integer is a sum of one or more numbers of the form 2^r3^s, where r and s are nonnegative integers and <u>no summand divides another</u>.
(For example, $23 = 9 + 8 + 6$.)

Pigeon Hole Principle (Putnam 1978)

Let A be any set of 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \cdots, 100$. Prove that there must be two distinct integers in A whose sum is 104.

Invariance/Monovariance

(Week 4) A bunch of children are playing in groups on a playground (a group might have only one child in it). Every minute, one child leaves their current group and joins a group that has at least as many children as their previous group. Prove that eventually all of the children are playing in one huge group.

Practice Problems

Problem 1: Induction

Show that for $n \geq 6$ a square can be dissected into n smaller squares, not necessarily all of the same size.

Problem 2: Pigeon Hole Principle

Five points are situated inside an equilateral triangle whose side has length one unit. Show that two of them may be chosen which are less than one half unit apart.

Problem 3: Invariance/Monovariance

[1] Several stones are placed on an infinite (in both directions) strip of squares. As long as there are at least two stones on a single square, you may pick up two such stones, then move one to the preceding square and one to the following square. Is it possible to return to the starting configuration after a finite sequence of such moves?

Some challenging problems

Problem A

[2] On an $n \times n$ board, there are n^2 squares, $n-1$ of which are infected. Each second, any square that is adjacent to at least two infected squares becomes infected. Show that at least one square always remains uninfected.

Problem B

[3] Seventeen numbers are chosen such that none of them have prime factors bigger than 10. Show that there exists 2 numbers in this list whose product is a perfect square.

Problem C

Show that a $2^n \times 2^n$ chess board with the bottom right square taken out can be tiled by L-shaped pieces made of 3 squares: a 2×2 grid with one corner piece removed.

[1] Problem 3 is from Yufei Zhao's MIT Putnam Course

[2] Problem A is from Stanford's Putnam Training 2007

[3] Problem B is from Problem Primer for the Olympiad